EXPONENTIAL AND LOG FUNCTIONS

Use this sheet to help you:

- Understand and apply the log laws
- Understand the application of the natural exponential and logarithmic functions
- Find the derivative of the natural exponential and logarithmic functions
The exponential function $y = 10^x$

$y = 10^x$ is an example of an exponential function.

The key feature is that the independent variable, $x$, appears as the power (exponent) to which a constant (in this case 10) is raised. The base does not need to be restricted to 10, in general, $y = a^x$ where $a$ is any constant greater than 1.

The inverse function to $y = 10^x$

Given the function $y = 10^x$
The inverse function is defined as $x = \log_{10}y$.
To comply with convention we interchange the labels of the two variables and write the function as

$y = \log_{10}x$

The following table gives a comparison of the logarithmic and index laws.

<table>
<thead>
<tr>
<th>Log law</th>
<th>Index law</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log(mn) = \log m + \log n$</td>
<td>$x^m \times x^n = x^{m+n}$</td>
</tr>
<tr>
<td>$\log(m) = \log m - \frac{\log n}{n}$</td>
<td>$\frac{x^m}{x^n} = x^{m-n}$</td>
</tr>
<tr>
<td>$\log 1 = 0$</td>
<td>$x^0 = 1$</td>
</tr>
<tr>
<td>$\log_a a = 1$</td>
<td>$x^1 = x$</td>
</tr>
<tr>
<td>$\log_a m = \frac{\log m}{\log a}$</td>
<td></td>
</tr>
</tbody>
</table>

Practical uses of logs

Given $3^x = 91$, we want to find the value of $x$.
We cannot just use the $y^x$ button on our calculator, as this can only be used when we know the value of $x$.
The method of solution is to take logs on both sides of the equation.
$log(3^x) = log 91$, now from our log rules $log(3^x) = x log 3$
$x log 3 = log 91$
dividing both sides by $log 3$ gives (using the log key on your calculator)
$x = \frac{log 91}{log 3} = 1.9590 = 4.1061$

This type of problem arises in economics when we want to solve for the number of years, $x$, in the compound growth formula $y = a(1 + r)^x$. 
The natural exponential function

\[ y = e^{x} \]

Now as \( e \) is a constant, this is an exponential function. It is called the natural exponential function, because it was first developed to describe growth in nature, where plants and animals grow continuously.

**Natural logarithms**

\( y = \log_{e}x \) is the natural logarithm function. It is commonly written as \( y = \ln x \).
The same rules that applied to common logs also apply to natural logs.

**Derivatives of exponential and logarithmic functions**

The derivative of the natural exponential function is

If \( y = e^{x} \), then \( \frac{dy}{dx} = e^{x} \)

If \( y = e^{f(x)} \), then \( \frac{dy}{dx} = e^{x} \cdot f'(x) \), where \( f'(x) \) denotes the derivative of \( f(x) \).

**Example 1:** if \( y = e^{3x + 1} \), where \( f(x) = 3x + 1 \), so \( f'(x) = 3 \)
then \( \frac{dy}{dx} = 3e^{3x + 1} \)

**Example 2:** if \( y = e^{-0.5x^{2}} \), where \( f(x) = -0.5x^{2} \),
so \( f'(x) = -x \)
then \( \frac{dy}{dx} = -xe^{-0.5x^{2}} \)

The derivative of the natural logarithm function is

If \( y = \ln x \), then \( \frac{dy}{dx} = \frac{1}{x} \)

If \( y = \ln f(x) \), then \( \frac{dy}{dx} = \frac{f'(x)}{f(x)} \)

**Example 1:** if \( y = \ln (x^{3} + 2x) \), where \( f(x) = x^{3} + 2x \),
so \( f'(x) = 3x^{2} + 2 \)
then \( \frac{dy}{dx} = 3x^{2} + 2 \)

**Example 2:** if \( y = 2\log_{e}x \)
then \( \frac{dy}{dx} = 2 \frac{1}{x} \)