THE UNIVERSITY OF MELBOURNE

FACULTY OF BUSINESS & ECONOMICS

Helpsheet

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INTEGRATION

Use this sheet to help you:

- Recognise an indefinite integral and evaluate indefinite integrals in simple cases
- Understand the formula for finding a definite integral and apply it in simple cases
- Apply integration to derive total cost from marginal cost

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The indefinite integral

The fundamental theorem of the indefinite integral says that integration (the process of evaluating an indefinite integral) is the reverse of differentiation. Because integration reverses differentiation, when we integrate we have to re-introduce the additive constant that is lost when we differentiate.

Integration Rules

Power rule of integration

$$\int x^n dx = \frac{x^{n+1}}{(n+1)} + C, \quad n \neq -1$$

C is an arbitrary constant

e.g. $\int x^3 dx$

$n = 3, so$

$$\frac{x^{n+1}}{n+1} = \frac{x^{3+1}}{3+1} = \frac{x^4}{4} + C$$

Multiplicative constants

$$\int ax^n dx = a\int x^n dx = a\frac{x^{n+1}}{(n+1)} + C, n \neq -1$$

e.g. $\int 10x^{0.25} dx$

$a = 10, n = 0.25$, so we get

$$10\int x^{0.25} dx = 10\left(\frac{x^{0.25+1}}{0.25+1}\right) + C$$

$$= 10\left(\frac{x^{1.25}}{1.25}\right) + C = 8x^{1.25} + C$$

Sums or differences

$\int f(x) + g(x) = \int f(x) + \int g(x)$

e.g. $\int [2x + 3x^2]dx = \int [2x]dx + \int [3x^2]dx$

$$= \frac{2x^2}{2} + \frac{3x^3}{3} + C$$

$$= x^2 + x^3 + C$$

Function of a function

$$\int \{ f'(x). [\{(f(x)\}^n]\} dx = \frac{[f(x)]^{n+1}}{(n+1)} + C, n \neq -1$$

e.g $\int (3x^2 + 3)(x^3 + 3x)^3 dx$

since $3x^2 + 3$ is the derivative of $x^3 + 3x$, we have the pattern necessary to apply the function of a function rule,

$$\therefore \int (3x^2 + 3)(x^3 + 3x)^3 dx = \frac{(x^3 + 3x)^4}{4} + C$$

Product and quotients

There is a single rule of integration which covers both products and quotients, and thus reverses the product and quotient rules of differentiation. It is called integration by parts. It is quite a difficult rule to apply and it doesn't always work. As it is not often needed in economics, we will not look at it.

The exponential function

$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

e.g. $\int [2xe^{x^2}]\,dx$

$= \frac{2xe^{x^2}}{2x} + C = e^{x^2} + C$

The natural logarithmic function

$\int \frac{1}{x}\,dx = \log_e x + C$ for $x > 0$ ($\log_e x = \ln x$)

and $\int \left[\frac{f'(x)}{f(x)}\right] dx = \ln f(x) + C$

note: the power rule of integration must not be applied when x is raised to the power -1.

e.g $\int \left[\frac{3x^2}{x^3+1}\right] dx$, since $f'(x)$ of $x^3 + 1 = 3x^2$,

then $\int \left[\frac{3x^2}{x^3+1}\right] dx = \ln(x^3+1) + C$

Finding a definite integral

The fundamental theorem of the definite integral

If $\int [f(x)]\,dx = F(x) + C$

that is, F(x) is the indefinite integral of f(x), then:

$\int_a^b [f(x)]dx = [F(x)]_{(x=b)} - [F(x)]_{(x=a)}$

where [F(x)](x=b) denotes the value of the function F(x) when x = b, and similarly for x = a.
e.g. Evaluate the definite integral of: $[x^2]dx$

$$\int_2^4 [x^2]dx = \left[\frac{x^3}{3}\right]_{x=4} - \left[\frac{x^3}{3}\right]_{x=2}$$

$$= \frac{64}{3} - \frac{8}{3} = \frac{56}{3} = 18⅔$$

Economic applications

1. Deriving the total cost (TC) function from the marginal cost (MC) function

If we are given a marginal cost function MC = f(q), by finding the indefinite integral we will obtain the TC function. This is known as the MC function is obtained by finding the derivative of the TC function. It is known that the reverse process of finding a derivative (differentiation) is integration.

As we are finding an indefinite integral an arbitrary constant, c, will appear in our solution. Unless we are given more information we are not able to evaluate this constant. In economic terms the integral (without the constant) we have found represents the variable costs, whilst the constant, c, represents the fixed costs. Thus from the information given we can find the total variable cost, but we cannot find the total fixed cost without further information.

2. Deriving total revenue (TR) from the marginal revenue (MR) function

Similarly to the previous section, if we are given a marginal revenue (MR) function, we can use integration to work backwards and recover the total revenue (TR) function from which the MR function was derived. Once we have the TR function it is only a small step to the demand function. In the MC function we interpreted c as fixed costs, but there can be no constant term in a total revenue function because when q = 0, nothing is sold, so there is no revenue. Therefore the TR function must pass through the origin. This additional information, derived from the economics of this case, means that we can set the arbitrary constant, c, equal to zero.

Some other economic applications using integration are:

a) consumers' surplus
b) producers' surplus
c) present value of a continuous stream of income.