THE UNIVERSITY OF MELBOURNE
FACULTY OF BUSINESS & ECONOMICS

Helpsheet

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CRAMER'S RULE

Use this sheet to help you:

- Use an alternative method to solve a set of linear simultaneous equations by matrix methods

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CRAMER'S RULE

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Cramer's rule is an alternative way of solving a set of simultaneous linear equations by matrix methods.

Suppose we have three simultaneous linear equations:
$4x + y - 5z = 8$
$-2x + 3y + z = 12$
$3x - y + 4z = 5$

If we define

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ \boldsymbol{b} = \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}$$

then we can write the simultaneous equations as:
$Ax = b$

Cramer's rule says that to find the first unknown, x, in the vector x of unknowns, we proceed as follows.

Step 1.

Take the matrix A and form a new matrix by replacing the first column of A with the column vector b. We call this new matrix A_1. So A_1 is given by:

$$\mathbf{A}_1 = \begin{bmatrix} 8 & 1 & -5 \\ 12 & 3 & 1 \\ 5 & -1 & 4 \end{bmatrix}$$

Step 2.

Calculate the determinants $|A|$ and $|A_1|$. Given $|A| = 98$ (from an example on the matrix algebra help sheet). Using the first row of A_1, we can calculate $|A_1|$ as

$$|A_1| = 8[(3 \times 4) - (-1 \times 1)] - 1[(12 \times 4) - (5 \times 1)] + (-5)[(12 \times -1) - (5 \times 3)]$$
$$= 196$$

Step 3.

The first unknown, x, in the vector x of unknowns, is then given by

$$x = \frac{|A_1|}{|A|} = \frac{196}{98} = 2$$

To find the second unknown, y, in the vector x of unknowns, we follow steps 1 to 3 above, but in step 2 we form a new matrix $|A_2|$, by replacing the second column of A with the column vector b. So A_2 is given by

$$\mathbf{A}_2 = \begin{bmatrix} 4 & 8 & -5 \\ -2 & 12 & 1 \\ 3 & 5 & 4 \end{bmatrix}$$

And in step 3 we find that:

$$y = \frac{|A_2|}{|A|} = \frac{490}{98} = 5$$

Finally, the third variable, z, is found in the same way by forming a new matrix, A_3, by replacing the third column of A with the column vector b. The solution for z is then:

$$z = \frac{|A_3|}{|A|} = \frac{98}{98} = 1$$

This gives us the solution, x = 2, y = 5 and z = 1

A set of simultaneous equations can be solved either by matrix inversion or by applying Cramer's rule. The choice between the two is a matter of personal choice, though Cramer's rule has the advantage that less tedious computations are necessary if we are only interested in the solution value of one of the variables.

In practice however computer programs such as Excel, can both evaluate determinants and invert matrices.

A macroeconomic application

As a simple illustration of the use of matrix algebra in macroeconomics, we can start from a generalised macroeconomic model.

$Y = C + I$ (equilibrium condition)
$C = aY + b$ (consumption function, a behavioural relationship)
$I = \bar{I}$ (investment, assumed exogenous) (1)

We have simplified our model by assuming $\hat{C}$ (planned consumption) always equals actual consumption, C.
Government's tax revenue, T, is given by
$T \equiv tY$ (an identity) (2)

Because of taxes, we must distinguish between income before deduction of taxes, Y, and disposable income, Y_d, defined as income net of taxes. So we have
$Y_d \equiv Y - T$ (an identity) (3)

And consumption now depends on disposable income so now,
$C = aY_d + b$ (4)

Finally the government's good and services, G, constitutes another component of aggregate spending, so to incorporate this
$Y = C + I + \bar{G}$ (5)

Our model is now complete and consists of the five equations labelled (1) – (5). However we can reduce this to three equations by using (1) to substitute for I in (5) and using (3) to substitute for Y_d in (4). Our reduced form is:
$T = tY$ (6)

$C = a(Y - T) + b$ (7)

$Y = C + \bar{I} + \bar{G}$ (8)

Before using matrix algebra on this set of three simultaneous equations, it is convenient to rearrange them slightly as:

$Y - C = \overline{I} + \overline{G}$ (9)

$-aY + C + aT = b$ (10)

$-tY + T = 0$ (11)

The reason for doing this, is that it puts all the exogenous variables, I, G and b on the right-hand side. We can now write these three equations in matrix form as:

$$\begin{bmatrix} 1 & -1 & 0 \\ -a & 1 & a \\ -t & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} \overline{I}+\overline{G} \\ b \\ 0 \end{bmatrix}$$

We can denote this as:

Ax = b

Where A is a matrix of parameters (the marginal propensity to consume, a, and the tax rate, t), x is a vector of unknowns and b is a vector of exogenous variables.

We can solve this set of equations by finding $\mathbf{A^{-1}}$, for then we can write:

$\mathbf{x = A^{-1}}$

To find A^{-1} follow the procedure explained on the Matrix Algebra Helpsheet. The matrix of cofactors is:

$$C = \begin{bmatrix} \begin{vmatrix} 1 & a \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -a & a \\ -t & 1 \end{vmatrix} & \begin{vmatrix} -a & 1 \\ -t & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ -t & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ -t & 0 \end{vmatrix} \\ \begin{vmatrix} -1 & 0 \\ 1 & a \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -a & a \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -a & 1 \end{vmatrix} \end{bmatrix}$$

When we multiply out all the 2 x 2 determinants, this becomes

$$C = \begin{bmatrix} 1 & a(1-t) & 1 \\ 1 & 1 & t \\ -a & -a & 1-a \end{bmatrix}$$

which transposed is:

$$C' = \begin{bmatrix} 1 & 1 & -a \\ a(1-t) & 1 & -a \\ 1 & t & 1-a \end{bmatrix}$$

Next find the determinant of A $|A|$. Using row 1 we get

$$|A| = 1\begin{vmatrix} 1 & a \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -a & a \\ -t & 1 \end{vmatrix} + 0\begin{vmatrix} -a & 1 \\ -t & 0 \end{vmatrix}$$

$$= 1 - a + at = 1 - a(1-t)$$

Finally we have

$$A^{-1} = \frac{1}{|A|}C' = \frac{1}{1-a(1-t)} \begin{bmatrix} 1 & 1 & -a \\ a(1-t) & 1 & -a \\ 1 & t & 1-a \end{bmatrix}$$

So the solution to the set of simultaneous equations is
x = A^{-1}b, that is:

$$\begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \frac{1}{1-a(1-t)} \begin{bmatrix} 1 & 1 & -a \\ a(1-t) & 1 & -a \\ 1 & t & 1-a \end{bmatrix} \begin{bmatrix} \overline{I}+\overline{G} \\ b \\ 0 \end{bmatrix}$$

The attraction of the above equation is, first we can quickly extract the solution value of any of the unknowns. e.g., by multiplying out the first row and column on the right-hand side we get:

$$Y = \frac{1}{1-a(1-t)}(\overline{I} + \overline{G}) + \frac{1}{1-a(1-t)}b$$

Second, we have our solution in a completely general and transparent way, so if any exogenous variable or a parameter changes in value, we can immediately calculate the effect on the unknowns. Finally, and perhaps most importantly, this method of solution can easily accommodate an increase in the number of equations and unknowns. This is vital as in real world macroeconomic models used by government economists and others for analysis and forecasting may well consist of several hundred equations.